Using Algebra to Solve Problems: Selecting, Symbolising, and Integrating Information

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We report an investigation of students' attempts to formulate equations for word problems. Ninety students in Years 9, 10 and 11 were tested twice over 10 months. We trace a progression of stages from naming quantities through describing relationships to writing equations and solving them. Even when all relationships were recognised and correctly symbolised, integrating them into an equation was a common difficulty.

Over the past two years we have been investigating students' performance in using algebra to solve word problems. In Stacey and MacGregor (1995) we reported our findings concerning the effects of problem presentation on students' solution methods. We found that a large number of students, concentrated in particular schools, had not attempted algebraic methods in spite of the instruction to write an equation for each problem and solve it. Their solution methods, usually successful, included informal arithmetic, guess-check-improve procedures, and complex reasoning. However there were many others who had used algebraic symbolism to name quantities, to label parts on a diagram, or to express relationships in a problem, but who did not go on to combine these pieces of information as an equation. To solve the problem they switched to arithmetic calculation or a guess-check-improve procedure. One school where the majority of students were in this category allowed us to use the same test items some months later with the same students. The test results indicate that on the final test most of these students still did not formulate a correct equation and solve it. By comparing their written responses in the two tests, we were able to see how they approached the task of formulating an equation for a problem, in what ways they had improved, and what the difficulties were that prevented success.

Method

Subjects

The students were in Years 9 and 10 (age 14-15) for the initial test, and Years 10 and 11 (age 15-16) for the final test. All classes involved were mixed-ability classes. The school is a coeducational school in a middle-class suburb.

Test

The test comprised six algebra word problems varying in difficulty. Students were asked to write an equation for each problem and solve it. In Figure 1 we show three of the problems, which students in Years 10 and 11 should be able to solve, and we refer to them in our discussion.

Procedure

The testing was administered by the mathematics teachers at the school, and carried out as part of normal lessons. Completed test papers were sent to the researchers for marking and analysis. The first test was given in August 1994 and the second test was given 10 months later in June 1995.

WORK OUT THE ANSWERS BY WRITING EQUATIONS AND SOLVING THEM

1. A group of scouts did a 3-day walk on a long weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. The total journey was 80 km. How far did they walk on Saturday?

2. Jeff washes three cars. The second car takes 7 minutes longer than the first one. The third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?

3. The three sides of a triangle are different lengths. The second side is 3 cm longer than the first side, and the third side is twice as long as the first side. The side lengths add up to 63 cm altogether. How long is the first side?

Figure 1. Test items

Results

Test papers showing all written working and answers were obtained from 90 students who had done both tests. Their progress, judged on all six problems, is shown in Table 1. Students' best performance is represented. For example, if a student used algebraic letters to record given information in only one item of a test and used no algebra at all in the other items, this student is counted in the "Partial Use" category for that test.

In Table 1, "No algebra" means that there was no attempt to use any algebraic notation. "Partial Use" means that the student had attempted to use some algebraic notation, even if only to denote an unknown quantity by a letter (e.g., labelling a diagram for Problem 3 with the letter x on one side). Examples of partial use include a wide range of responses such as $x + 7 + x + 13 = 80$ (Problem 1) in which one of the parts is not represented, and correct equations in three variables such as $x + y + z = 80$ which were not further used to solve the problem. "Equation" means that a correct equation was written. By "correct equation" we mean an equation in one variable that could have been used, or was used, to solve the problem (e.g., $x + x + 3 + x + x = 63$ for Prob.3).

Table 1
Progress in students' use of algebra over 10 months (N = 90)

	Final test		
Initial test	No algebra	Partial use	Equation
No algebra	53	11	2
Partial use	2	9	8
Equation	0	0	5
TOTAL	55	20	15

For each problem, correct numerical answers were written by approximately 65%

of students in the first test and 75% in the second test. Most answers to all the problems, whether right or wrong, were obtained by non-algebraic methods. As Table 1 shows, there were 35 students (39% of the sample) in the final test who attempted some algebra. Five students formed and solved equations correctly in all three tests. However three of the five did not use standard algebraic techniques for solving their equations. One used a systematic trial-and-error routine, trying different values for x in his equation. The others wrote informal calculations that were not necessarily based on their equations.

It might be argued that since algebraic methods are not necessary for solving the problems, students who were capable of using algebra may not have done so and therefore have been wrongly classified in the "No algebra" category. However algebra was required for the harder problems in the test (see Appendix). The students who demonstrated their capability to use algebra for these harder problems had used it for all the problems, and therefore are correctly classified in the table as users of algebra.

We had expected that in the time between the tests many students would move from using no algebra to being able to formulate equations. However, as Table 1 shows (see first row), 53 students continued to make no attempt to use algebra, 11 moved from "No algebra" to "Partial use" and only two students moved from "No algebra" to writing a correct equation. The second row of the table shows that eight students moved from being partial users to writing an equation, and nine did not progress. There were 20 students at the time of the final test who used some algebraic notation but had not yet progressed to a stage where they could write an equation for these relatively straightforward problems. It is important to point out that five students who are seen in the table as "progressing", that is, moving from using no algebra to using some algebra notation, had solved the problems successfully in the first test but failed to solve them in the second test when they tried to use algebra. It is to be expected that students like these are reluctant to use algebraic methods and come to believe that algebra is unreliable and unnecessary.

Two students used some algebra in early tests but no algebra in the final test. Their apparent regression can be attributed to the effects of teaching. Their responses, and the responses of other students in their class, indicated that they had been trained to use a systematic trial-and-error approach to problem-solving. Although frequently tedious and time-consuming, this method usually resulted in correct answers. Training in using trial-and-error as the easiest or only way to solve a problem may be a reason why so few of the partial users of algebra did not progress, and why so many did not use algebra at all in either test.

Students attempted to deal with the information given in the problem in many different ways. Some of these were helpful ways of recording, whereas others contained specific well-documented errors that obstructed progress towards writing an equation. Stages in this progression, and examples of difficulties at each stage, are shown below. All the examples are chosen from students' written responses to Problem 1, unless stated otherwise.

1. Naming the Unknown Quantities Referred to in the Problem

The use of letters as abbreviated words was a cause of difficulty for three students. For example, to begin Problem 3 one student wrote:

$$3 \text{ car} = 1c + 11$$
$$2c = 1c + 7$$
$$3c = 87$$

He has written "3 car" to mean "time taken for the *third car*", "$1c$" to mean "time taken for the *first* car", and $3c$ to mean "total time for *three cars*". He understands the relationships in the problem situation, but has not understood the crucial difference between letters as abbreviated words and letters as representing quantities or variables. It is widely recognised that in certain circumstances students tend to use algebraic letters as shorthand names for objects or as abbreviated words (Booth, 1986). As this example demonstrates, the use of c to mean the word *cars* blocked the route to an algebraic solution.

2. Expressing the Relationships Between the Parts

Many students were able to express the relationships in the problem situations, often in correct algebraic notation (e.g., x, $x + 7$, $x + 13$). For Problem 3, diagrams were frequently drawn and labelled appropriately (e.g., x, $x + 3$, $2x$). However there were several students who used a different letter for each unknown quantity (e.g., X, Y, Z) and did not go on to express these in terms of a single variable. Whether one or more letters were used, students who did not write an equation abandoned any further use of algebra and switched to non-algebraic methods for solving the problems.

Several students made errors in notation that blocked further use of an algebraic method. These errors included concatenation for addition and exponential notation for a product. Concatenation for addition was seen in the work of four students; for example, $x7$ and $x13$ to mean "7 more than x" and "13 more than x". Exponential notation for a product was seen in the work of six students; for example, x^2 to mean "twice x".

These errors generally blocked further development of an algebraic procedure. However one student who wrote $x^3 + 20 = 60$ interpreted the exponential notation as a product and solved the equation. Summarising the data above, we see that of the 37 students who attempted to use some algebra, 10 students (approximately one-quarter of the sample) were prevented from writing correct equations by their misunderstandings of algebraic notation for sums and products. Their misuse of notation may indicate a poorly-developed concept of multiplication and its relationships with repeated addition and repeated multiplication (Stacey & MacGregor, 1994).

3. Writing a Useful Equation that Integrates the Problem Information

Some students wrote correct equations or expressions for all four relationships but did not combine them into an equation in one variable. For example, for Problem 3 a student labelled the three sides of her diagram of a triangle as

$$A$$
$$B = A + 3$$
$$C = 2 \times A$$

and then wrote the equation A + B + C = 63. She did not make any use of this equation and did not solve the problem. Another student wrote, for the same problem,

$$S1 = x,\ \ S2 = x + 3,\ \ S3 = 2x$$

Although this also looks a useful beginning, where S1 means "Side 1" and correct expressions in terms of x have been written for each side of the triangle, the student wrote as his equation S1 + S2 + S3 = 63. He then abandoned any further use of algebra and solved the problem by a trial-and-error method.

Several students did not know how to write equations in the standard way, although they appeared to understand what relationships were involved and integrated them to some extent. Examples are shown in Figure 2. They were able to solve the problems by non-algebraic methods, and perhaps had used their "equations" in some way to guide reasoning and calculation. It seems likely that with appropriate instruction they would quickly learn how to write equations in the standard way. It is puzzling why they had not learned to take this small last step.

4. Equations as Descriptions of Procedures Used for Calculating

Several students calculated answers to each problem by arithmetic reasoning, and then tried to represent these calculations as equations. This method of dealing with algebra word problems has been observed by other researchers (Arzarello, Bazzini & Chiappini, 1993). These equations were not representations of problem structure, but descriptions of the procedure used to calculate a value for one of the unknowns. For example, a student wrote the equation for Problem 2 as

$$x = (87 - 18) \div 3$$

and others wrote

$$x = 87 - 18 = 69 \div 3 = 23$$

where x stands for the time taken to wash the first car. It can be argued that, technically, the first of these is an acceptable equation. However in the harder problems on the test, which were too difficult to solve by mental reasoning and arithmetic, students who were limited to writing a description of the solution method had no chance of success. Lacking the support of an algebraic representation of the problem, they were unable to devise a solution.

(i) Sat Sun Mon

$x \rightarrow x + 7 \rightarrow x + 13 = 80$

(ii)

x

$x + 7$

$\underline{x + 13}$

80

(iii)

$1 = y$

$2 = y + 7$

$3 = y + 13$

$1 + 2 + 3 = 80$

(iv) $x,\ x + 7,\ x + 13,\ 80$

(v) <---------------------- 80 ------------------------------>

$a = ?\ b = a + 7 \qquad c = a + 13$

Figure 2. Attempts to integrate problem information

Discussion

Our data indicate that major difficulties in formulating equations in the test did not lie in students' failure to comprehend the written information, to understand the problem structure, or to see how the parts were related to each other and to the whole. Most students could solve the problems by non-algebraic methods, providing confirmation that understanding the problem situation was not a difficulty. For the students who tried to use algebra, the main obstacles to success were (a) incorrect use of algebraic syntax, and (b) failure to integrate the given information as an equation or set of equations. There were others who wrote correct equations that could be used to solve the problems, but did not use these equations, apparently not knowing how to use the notation as a tool for deductive reasoning.

As Arzarello et al. (1993) have commented, writing equations for problems is a complex process. These researchers have suggested that the process of naming (i.e., the choice of variables) and understanding the main relations in the problem are the most crucial steps. Our data indicate that, for most students in the sample, naming variables and understanding relations were not difficult for the simple problems we used. Most students who tried algebra could name quantities, and there was little difficulty related to expressing several quantities in terms of one variable. However there were several instances of students who named the three parts in a problem appropriately (e.g., x, $x + 7$, $x + 13$) but did not try to relate them to the total. As we have shown, some students used unconventional formats such as arrow-diagrams, vertical addition, or invented notations to try to denote the idea that the sum of the parts is equal to the total given. They were unable to write an equation to express the structure of the problem situation. Others had not learned that an equation is written to represent the problem situation; they wrote a description of the calculation procedure they had used to solve the problem. These students have perceived the equation as a formula for calculating. They need to know that algebra can also be used to extend and support logical reasoning; its purpose in problem-solving is not to describe a solution procedure that has already been constructed mentally.

In a typical school algebra curriculum, the first problems given to students to solve by algebraic means can also be solved by simple arithmetic, intuitive reasoning, or a simple guess-and-check. Until they achieve a certain level of fluency, students see algebra as an extra difficulty or unnecessary task imposed by teachers for no obvious purpose and not as a useful tool for making problem-solving simpler. This attitude is reasonable, since the problems they have so far encountered (such as the three problems presented in this paper) are not good examples of the power of algebra. Cortes, Vergnaud & Kavafian (1990) state that if students are to learn to formulate algebraic equations for solving problems, teachers need to discourage the search for non-algebraic solutions. We support this view, while reminding readers that it is difficult to find problems that are sufficiently complex to warrant an algebraic solution but easy enough for students to work through with understanding and learn from.

It is generally assumed that when comprehending a mathematical problem and preparing to solve it people construct a mental representation of some kind. Several theories have been proposed about the form and function of these representations. Using the context of elementary-grade arithmetic problems, Kintsch and Greeno (1985) proposed that a problem representation is built in several steps, beginning with a conceptual representation of meaning in the form of a set of propositions (called a *propositional text base* by Kintsch and Greeno). Johnson-Laird (1983) also proposed that the initial stage of comprehension produces a propositional representation. The individual combines this propositional representation with other general and specific knowledge to construct the second representation - an integrated and articulated mental model of the problem situation. We suggest that the initial set of propositions (Johnson-Laird's *propositional representation*) is sufficient for solving problems by the guess-and-check method. The propositional representation provides comprehension of each piece of information given without necessarily relating it to other information in a single model. The problem-solver guesses a value for the unknown, and checks whether it allows each proposition to be true. The process is essentially substitution in expressions. Almost all the problems we have seen in textbooks for the first four years of school algebra can be solved quickly by this method because of the nature of the numbers involved, and consequently many students use it. Students who have a good number sense often reach the answer after three or four guesses. An algebraic equation does not represent the separate pieces of propositional information but arises from the integrated mental model that is produced in the later stage of comprehension. As we have seen, some students do not know how to represent this model as an equation.

In their investigation of algebra word-problem comprehension, Nathan, Kintsch and Greeno (1992) have pointed out that students may understand a problem in everyday terms but be unable to represent its formal aspects as required for an algebraic solution. These researchers suggest that features of the student's cognitive representation of the problem determine what information is available for reasoning. Our data provide no evidence that students' mental models were inadequate or incorrect. The students demonstrated their capabilities in comprehension, logical reasoning, written calculation, mental arithmetic, and problem-solving by non-algebraic methods. There is no obvious explanation for their difficulties in constructing equations for the simple problems they were given. It seems likely that they had not had sufficient experience. We conclude that formulating an equation is not an intuitive way to represent a problem, but needs to be carefully taught.

Reluctance to formulate an equation may be one indicator of the gap between arithmetic and algebraic thinking that is now widely recognised in the literature. Cortes, Vergnaud & Kavafian (1990, p. 28) have pointed out that "most pupils are not familiar with the concept of the equation. For them, an equation is an abbreviated way of writing the terms of the problem: a summary". Our data support this assertion. In certain contexts students are familiar with equations as formulas, that is, as instructions for arithmetic calculation. They know how to use a formula by substituting values and calculating the answer. As Cortes et al. observed, "this tool-like characteristic of the equation" (p. 28) when written as a formula (e.g., $c = ax + b$) is not visible to students when the equation is written as $ax + b = c$. The concept of an equation as a statement about relationships, rather than as a formula, may be crucial to students' ability to use algebra for solving problems. Students' perceptions of what an equation represents and how it relates to a problem are being investigated in the current stage of our project.

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Appendix

The Two Hard Problems in the Test

(i) I think of a number, multiply it by 8, subtract 3, and then divide by 3. The result is twice the number I first thought of. What was the number?

(ii) Chris runs from A to B, a distance of x Km, and immediately cycles back along the same route to A. The total time to go there and back is 3 hours. The running speed is 10 Km/hr and the cycling speed is 40 Km/hr. Write an equation using the fact that the total time is 3 hours. Solve your equation to find the value of x.